The data comparisons were similar to the previous cases. The normalized skin friction increased by a factor of 1.7 over its zero pressure gradient value and the conventional eddy viscosity model again considerably underpredicted this increase. Velocity profiles were compared and again good agreement with the data obtained. The agreement deteriorated slightly as the flow proceeded downstream; however, the revised model, in all cases, was in considerably better agreement with the data than the conventional model. A summary of the revised constants used in the pressure gradient region for the three values of Reynolds number is given in Table 1.

The value of k_1 increased to 0.65 immediately upon encountering the pressure gradient, as in Laderman's case, independent of streamwise distance and Reynolds number within the limited range of values of these test parameters. The damping constant A^+ appears to be independent of streamwise distance and not independent of Reynolds number, while the Clauser constant k_2 increases with streamwise distance.

C. Waltrup and Schetz's Experiment

Unlike the previous cases, Waltrup and Schetz[3] induced three pressure gradients on the flat wind-tunnel floor using sting-mounted compression surfaces along the tunnel centerline. These pressure gradients were not constant. The skin friction also increased for this experiment, characteristic of entering an adverse pressure gradient region, and poorer predictions of the conventional viscosity model were evident. The boundary-layer thickness was predicted as well by either viscosity model when the experimental thickness was taken as the point where $u = 0.995\ U_e$, in agreement with the definition used for the computations. The calculated velocity profiles were in good agreement in the outer wake portion; however, the conventional viscosity model performed as well as the revised model. Neither model predicted the experimental results in the inner or wall region. Various scaling factors and many values of k_1, A^+, and k_2 were tried, but agreement with these data points near the wall could not be obtained.

Conclusions

The constants in the algebraic eddy viscosity models are not constant for flow with adverse gradients. These calculations show that the inner layer reacts immediately through the constants k_1 and A^+ to the imposed pressure gradient, while the outer layer alone requires a lag equation (varying with streamwise distance) to permit the constant k_2 to attain its maximum value.

At present, a correlation of the values of k_1, A^+, and k_2 with physical properties of the flow has not been achieved. The values obtained to date are shown in Table 1. It is evident that additional experimental data will be necessary. Additional values of these constants, for various flows, may enable verification and extension of known correlations and rate equations or new correlations with a wider range of applicability to be determined. Alternatively, the scatter in these values may preclude correlations, in which case, the limitations of this model will have been defined and an advanced model will have to be tested.

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Limiting Particle Streamline of Gas Particle Mixtures in Axially Symmetric Nozzles

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Introduction

MUCH work has been done to estimate phase nonequilibrium effects on the performance of rocket engines. In general, however, an estimate of such an effect is very difficult, even in the quasi-one-dimensional approximation, and is often almost impossible, at least theoretically, for multidimensional flows. Much of the previous work, therefore, is primarily concerned with the quasi-one-dimensional flows.[1] There have been only a few studies of two-dimensional or axially symmetric nozzle flows.[2]

One of the complications in the gas particle flows arises because of the inertia of the solid particles. The particle streamlines may deviate from those of the gas, and their trajectories are determined by following the individual particles through the flowfield of the gas. Since the particles exert no pressure, particle trajectories near a nozzle wall are unaware of and uninfluenced by the presence of the wall boundary, suggesting that the particle streamlines near the wall are not required to be parallel to the wall. The particles may, therefore, detach from the wall or impinge on the wall. In the flows with particle detachment from the wall, there exists a particle-free region between the so-called limiting particle streamline and the wall boundary. The appearance of such a particle-free region in a flow introduces a mathematical difficulty. Also, the shape and location of the limiting streamline and the location of the point of the particle impingement on the wall are very important to the nozzle performance.

In this Note, particle behavior in axially symmetric supersonic nozzles is considered under the condition that the velocity and temperature lags are small. Particular attention is paid to the particle streamlines, especially limiting streamlines. The first-order problem is solved for particle streamlines.

Coordinate Systems and Basic Equations

The dimensionless quantities are introduced by

$$x=\bar{x}/\bar{L},\ y=\bar{y}/\bar{r},\ \rho=\bar{\rho}/\bar{\rho}_*,\ p=\bar{p}/\bar{p}_*,\ V=\bar{V}/\bar{a}_*,\ T=\bar{T}/\bar{T}_*$$

$$\rho_p=\bar{\rho}_p/\bar{\rho}_{p*},\ V_p=\bar{V}_p/\bar{a}_*,\ T_p=\bar{T}_p/\bar{T}_* \tag{1}$$

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where $\bar{x}$ is the distance along the nozzle axis measured from the throat, $\bar{y}$ is the radial distance from the $\bar{x}$ axis, and $\bar{\rho}, \bar{V}, \bar{p}, \bar{T}$, and $\bar{a}$ are the density, velocity, pressure, temperature, and speed of sound of the gas, respectively. The quantities associated with the solid particles of a single size are designated by a subscript p. The reference length $\bar{L}$ and $\bar{r}$ are the nozzle length and the half-height of the nozzle wall at the throat, respectively, and the asterisk denotes the reference sonic conditions. As we shall see, since the flows are treated as a perturbation from equilibrium, the reference sonic quantities with asterisk notation are evaluated for the equilibrium flow. Therefore, we have

$$Q_* = Q_{e*} \quad (Q = \rho, p, T, a, \text{ and } \rho_p) \tag{2}$$

where the subscript e denotes equilibrium conditions. In the equilibrium situation we have

$$\bar{a}_e^2 = \frac{\gamma_e}{1+\zeta}\frac{\bar{p}_e}{\bar{\rho}_e} = \frac{\gamma_e}{1+\zeta} R\bar{T}_e \tag{3}$$

where R is the gas constant, ζ is the ratio of the mass flow rate of the particles to that of the gas, and γ_e is the constant defined by

$$\gamma_e = \gamma[1+\zeta(C_{pp}/C_{pg})]/[1+\gamma\zeta(C_{pp}/C_{pg})] \tag{4}$$

In this equation, γ is the ratio of the specific heat of the gas, and C_{pg} and C_{pp} are the specific heat at constant pressure of the gas and the specific heat of the solid material.

For the theoretical analysis of the multidimensional flows, it is convenient to use the generalized coordinates of the streamline and the orthogonal trajectory to the streamlines

$$\alpha = \text{const along } \frac{dy}{dx} = -k\cot\theta \tag{5}$$

$$\beta = \text{const along } \frac{dy}{dx} = k\tan\theta \tag{6}$$

where

$$k = \bar{L}/\bar{r} \tag{7}$$

and θ is the flow angle of the gas relative to the nozzle axis. Also, similar variables (ξ, η) for the particles are introduced by

$$\xi = \text{const along } \frac{dy}{dx} = -k\cot\theta_p \tag{8}$$

$$\eta = \text{const along } \frac{dy}{dx} = k\tan\theta_p \tag{9}$$

Taking (α, β) as the independent variables to the system, we can consider that x, y, ξ, and η, as well as other flow quantities, are the functions of these variables.

Introducing the usual assumptions, the gas particle nozzle flow equations may be written as:

$$\frac{1}{\rho V}\frac{\partial}{\partial\alpha}(\rho V) + k\frac{(\partial x/\partial\alpha)}{(\partial y/\partial\beta)}\frac{\partial\theta}{\partial\beta} + \frac{1}{y}\frac{\partial y}{\partial\alpha} = 0 \tag{10}$$

$$\rho V\frac{\partial V}{\partial\alpha} + \frac{1+\zeta}{\gamma_e}\frac{\partial p}{\partial\alpha} + \zeta\frac{(\partial x/\partial\alpha)}{(\partial x/\partial\xi)}\frac{\cos\theta_p}{\cos\theta} \times \frac{[V - V_p\cos(\theta-\theta_p)]}{[V\cos(\theta-\theta_p) - V_p]}\rho_p V_p\frac{\partial V_p}{\partial\xi} = 0 \tag{11}$$

$$\rho V^2\frac{\partial\theta}{\partial\alpha} + k\frac{1+\zeta}{\gamma_e}\frac{(\partial x/\partial\alpha)}{(\partial y/\partial\beta)}\frac{\partial p}{\partial\alpha} + \zeta\frac{(\partial x/\partial\alpha)}{(\partial x/\partial\xi)}\frac{\cos\theta_p}{\cos\theta}\frac{\rho_p V_p^3}{V}\frac{\partial\theta_p}{\partial\xi} = 0 \tag{12}$$

$$\frac{\partial}{\partial\alpha}\left(\frac{1+\zeta}{\gamma_e}\frac{C_{pg}}{R}T + \frac{1}{2}V^2\right) + \zeta\frac{\rho_p V_p}{\rho V}\frac{(\partial x/\partial\alpha)}{(\partial x/\partial\xi)} \times \frac{\cos\theta_p}{\cos\theta}\frac{\partial}{\partial\xi}\left(\frac{1+\zeta}{\gamma_e}\frac{C_{pp}}{R}T_p + \frac{1}{2}V_p^2\right) = 0 \tag{13}$$

$$p = \rho T \tag{14}$$

$$\frac{1}{\rho_p V_p}\frac{\partial}{\partial\xi}(\rho_p V_p) + k\frac{(\partial x/\partial\xi)}{(\partial y/\partial\eta)}\frac{\partial\theta_p}{\partial\eta} + \frac{1}{y}\frac{\partial y}{\partial\xi} = 0 \tag{15}$$

$$V_p\frac{\partial V_p}{\partial\xi} = \lambda\frac{(\partial x/\partial\xi)}{\cos\theta_p}[V\cos(\theta-\theta_p) - V_p] \tag{16}$$

$$V_p^2\frac{\partial\theta_p}{\partial\xi} = \lambda\frac{(\partial x/\partial\xi)}{\cos\theta_p}V\sin(\theta-\theta_p) \tag{17}$$

$$V_p\frac{\partial T_p}{\partial\xi} = \lambda\mu\frac{(\partial x/\partial\xi)}{\cos\theta_p}(T-T_p) \tag{18}$$

$$\frac{\partial y}{\partial\alpha} = k\tan\theta\frac{\partial x}{\partial\alpha} \tag{19}$$

$$\frac{\partial x}{\partial\beta} = -\frac{1}{k}\tan\theta\frac{\partial y}{\partial\beta} \tag{20}$$

$$\frac{\partial y}{\partial\alpha}\frac{\partial\eta}{\partial\beta} - \frac{\partial y}{\partial\beta}\frac{\partial\eta}{\partial\alpha} = k\tan\theta_p\left[\frac{\partial x}{\partial\alpha}\frac{\partial\eta}{\partial\beta} - \frac{\partial x}{\partial\beta}\frac{\partial\eta}{\partial\alpha}\right] \tag{21}$$

$$\frac{\partial x}{\partial\alpha}\frac{\partial\xi}{\partial\beta} - \frac{\partial x}{\partial\beta}\frac{\partial\xi}{\partial\alpha} = -\frac{1}{k}\tan\theta_p\left[\frac{\partial y}{\partial\alpha}\frac{\partial\xi}{\partial\beta} - \frac{\partial y}{\partial\beta}\frac{\partial\xi}{\partial\alpha}\right] \tag{22}$$

where λ and μ are the constants depending upon the kind of gas and the particles, of which the former is assumed to be

$$1/\lambda \ll 1 \tag{23}$$

and the latter is usually of order unity. Obviously, Eqs. (19-22) have been derived from Eqs. (5, 6, 8, and 9), respectively.

Zeroth-Order Solution

Under the condition of Eq. (23), it will be reasonable to expand all dependent variables as follows:

$$Q = Q^{(0)} + \frac{1}{\lambda}Q^{(1)} + \frac{1}{\lambda^2}Q^{(2)} + \ldots \tag{24}$$

where Q may be any one of $x, y, \xi, \eta, \rho, \rho_p, V, V_p, \theta, \theta_p, T$, and T_p. Substituting Eq. (24) into the basic equations and rearranging suitably, we have a zeroth-order problem:

$$x^{(0)} = \xi^{(0)} = x_e, \quad y^{(0)} = \eta^{(0)} = y_e, \quad p^{(0)} = p_e \tag{25}$$

$$Q^{(0)} = Q_p^{(0)} = Q_e\,(Q = \rho, V, T, \theta)$$

$$\frac{1}{\rho_e V_e}\frac{\partial}{\partial\alpha}(\rho_e V_e) + k\frac{(\partial x_e/\partial\alpha)}{(\partial y_e/\partial\beta)}\frac{\partial\theta_e}{\partial\beta} + \frac{1}{y_e}\frac{\partial y_e}{\partial\alpha} = 0 \tag{26}$$

$$\rho_e V_e^2\frac{\partial\theta_e}{\partial\alpha} + \frac{k}{\gamma_e}\frac{(\partial x_e/\partial\alpha)}{(\partial y_e/\partial\beta)}\frac{\partial p_e}{\partial\beta} = 0 \tag{27}$$

$$\frac{p_e}{\rho_e}=\frac{\gamma_e+1}{2}\left(1-\frac{\gamma_e-1}{\gamma_e+1}V_e^2\right) \tag{28}$$

$$p_e=\rho_e{}^{\gamma_e} \tag{29}$$

$$p_e=\rho_e T_e \tag{30}$$

$$\frac{\partial y_e}{\partial\alpha}=k\tan\theta_e\frac{\partial x_e}{\partial\alpha} \tag{31}$$

$$\frac{\partial y_e}{\partial\beta}=-k\cot\theta_e\frac{\partial x_e}{\partial\beta} \tag{32}$$

Obviously this system is mathematically equivalent to the classically ideal one. In the present paper, the zeroth-order problem is solved approximately by the scheme in Ref. 3 for the strip number $n=1$, since the emphasis is placed upon particle behavior. The results for x_e,y_e, V_e, and θ_e are written as

$$x_e=\alpha \tag{33}$$

$$y_e=\beta f(\alpha) \tag{34}$$

$$\left\{\left(\frac{2}{\gamma_e+1}\right)^{1/(\gamma_e-1)}\Big/V_e\left[1-\frac{\gamma_e-1}{\gamma_e+1}V_e^2\right]^{1/(\gamma_e-1)}\right\}^{1/2}=f(\alpha) \tag{35}$$

$$\theta_e=\frac{1}{2}y_e\frac{(V_e^2-1)}{\left[1-\dfrac{\gamma_e-1}{\gamma_e+1}V_e^2\right]}\frac{1}{V_e}\frac{\partial V_e}{\partial\alpha} \tag{36}$$

where $y=f(x)$ denotes the prescribed nozzle contour.

First-Order Solution for the Particle Streamline

Even in the first approximation, it is very difficult to obtain the complete solution. Fortunately, however, the problem is easily solved for the particle streamline. From Eqs. (17), (21), and (22), we can get, respectively,

$$\theta_s^{(1)}=\theta^{(1)}-\theta_p^{(1)}=V_e\frac{\partial\theta_e}{\partial\alpha}\frac{\cos\theta_e}{(\partial x_e/\partial\alpha)} \tag{37}$$

$$\frac{\partial\eta^{(1)}}{\partial\alpha}=k\frac{(\partial x_e/\partial\alpha)}{(\partial y_e/\partial\beta)}\theta_s^{(1)} \tag{38}$$

$$\frac{\partial\xi^{(1)}}{\partial\beta}=-\frac{1}{k}\frac{(\partial y_e/\partial\beta)}{(\partial x_e/\partial\alpha)}\theta_s^{(1)} \tag{39}$$

Since the flow far upstream of the throat is in equilibrium and the nozzle axis is the streamline of both the gas and the particles, we have

$$\eta^{(1)}=k\int_{-\infty}\frac{\cos\theta_e}{(\partial y_e/\partial\beta)}V_e\frac{\partial\theta_e}{\partial\alpha}\mathrm{d}\alpha \tag{40}$$

$$\xi^{(1)}=-\frac{1}{k}\int_0\cos\theta_e\frac{(\partial y_e/\partial\beta)}{(\partial x_e/\partial\alpha)^2}V_e\frac{\partial\theta_e}{\partial\alpha}\mathrm{d}\beta \tag{41}$$

where β and α are held constant during integration in Eqs. (40) and (41), respectively. The integral term in Eq. (40) may become positive or negative, depending upon the nozzle geometry. In the former case, there exists a limiting streamline in the flowfield; in the latter, impingement of some particles on the wall occurs. Considering, for example, a

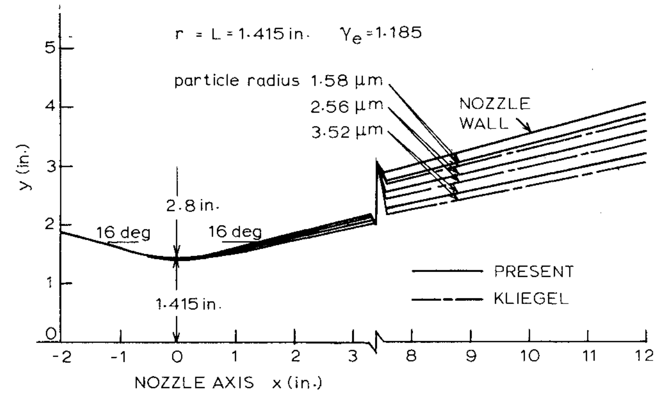

Fig. 1 Particle limiting streamline.

conical nozzle described by

$$f(x)=(1+Kx^2)^{1/2} \tag{42}$$

where K is a nozzle constant, we have a limiting streamline

$$1=\beta+\frac{1}{2\lambda}k\sqrt{K}\left(\frac{\gamma_e+1}{2}\right)^{1/(\gamma_e-1)}\beta\int_0^{V_e(\alpha)}F(V_e)\mathrm{d}V_e \tag{43}$$

where

$$F(V)=\frac{V^{5/2}\,|V^2-1|\left(1-\dfrac{\gamma_e-1}{\gamma_e+1}V^2\right)^{(5-2\gamma_e)/2(\gamma_e-1)}}{\left\{\left(\dfrac{2}{\gamma_e+1}\right)^{1/(\gamma_e-1)}-V\left(1-\dfrac{\gamma_e-1}{\gamma_e+1}V^2\right)^{1/(\gamma_e-1)}\right\}^{1/2}} \tag{44}$$

This comes from the fact that $0\leq\eta\leq1$ $(0\leq\beta\leq1)$ and $\eta=1$ denotes the streamline that coincides with the gas streamline along the wall ($\beta=1$) in the region far upstream of the throat.

Sample calculations of the particle limiting streamlines have been carried out for nozzle geometry shown in Fig. 1, and the results for $\gamma_e=1.185$ and for a few sizes of particles Al_2O_3 are shown in this figure being compared with the results by Kliegel.[2] This figure shows comparably good agreement with Kliegels's results for smaller particle sizes or large λ.

Although in the present work the first-order problem for the particle streamline has been solved with the approximate solution to the zeroth-order problem, it may be fairly accurate for large values of λ and for nozzles with slowly varying cross section, because $\eta^{(1)}$ depends strongly on the structure of the gas streamlines (x_e,y_e, and θ_e) in the zeroth-order approximation, which can be approximated very well by the quasi-one-dimensional results. Furthermore, it is important to note that $\eta^{(1)}$ has been given as an integral quantity which depends predominantly on the structure of the flowfield as a whole and depends little on the detailed local structure of the flowfield. This may perhaps be another reason for the good accuracy of the present results, since the quasi-one-dimensional solution usually can closely approximate the actual multidimensional flow as a whole.

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